Generalized Canonical Form of a Multi-Time Dynamical Theory and Quantization

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A generalized canonical form of multi-time dynamical theories is proposed. This form is a starting point for a modified canonical quantization procedure of theories based on a quantum version of the action principle. As an example, the Fokker theory of a direct electromagnetic interaction of charges is considered.

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I. INTRODUCTION

The well-known standard canonical quantization procedure is formulated for ordinary dynamical theories with a single parameter of time t, and the action of which may be written in a Lagrangian form:

$$I = \int L\left(q, \dot{q}\right) dt. \tag{1}$$

where $L\left(q, \dot{q}\right)$ is a Lagrangian function, the dot denotes the derivative with respect to the time parameter t. This procedure of canonical quantization originates from a canonical form of the classical theory. A canonical momentum for a dynamical variable $q_i, i = 1, 2, ..., n,$, where n is the dimensionality of a configuration space of a system, is defined as follows:

$$p_i = \frac{\partial L\left(q, \dot{q}\right)}{\partial \dot{q}_i}, \tag{2}$$

and then the Hamiltonian is defined by the Legendre transformation:

$$H\left(p, q\right) \equiv \left.\left(\sum_{i=1}^{n} p_i \dot{q}_i - L\left(q, \dot{q}\right)\right)\right|_{\dot{q}=\dot{q}(p,q)}, \tag{3}$$

where velocities are supposed to be excluded by use of Eq. (2). We do not consider here singular theories with constraints. This will be a subject of a subsequent work. The quantization (in a coordinate representation) is performed by the replacement of canonical variables (q_i, p_i) by operators acting on a wave function $\psi\left(q, t\right)$:

$$\begin{aligned} \widehat{q}_k \psi &\equiv q_k \psi, \\ \widehat{p}_k \psi &\equiv \frac{\hbar}{i} \frac{\partial \psi}{\partial q_k}. \end{aligned} \tag{4}$$

However, a quantization of a multi-time dynamical theory as the Fokker theory of direct electromagnetic interaction of charges [1], has a practical interest. In this theory a point charge e_a has its own time parameter t_a, and the action can not be written in the Lagrangian form (1). In the preprint series [2, 3] a new form of quantum mechanics in terms of a quantum action principle was proposed. It was noted that the new approach is the most appropriate for multi-time dynamical theories. To perform the modified canonical quantization procedure in that case, one needs a generalized canonical form of the action. In the present work such a form of the action for a multi-time dynamical theory is proposed. As an example, the Fokker theory of a direct electromagnetic interaction of two charges is considered.

II. MULTI-TIME DYNAMICAL THEORY

Let the dynamics of a system be described by a several number of sets of dynamical variables $q_a\left(t_a\right)$ with an own for each set time parameter $t_a \in [0.T_a]$. Numerating indices of dynamical variables in each set are omitted for brevity. We consider each set of dynamical variables q_a as coordinates of a particle in a configuration space. Let the action of that system be a smooth functional of coordinates of particles and their velocities:

$$I = I\left[q_a\left(t_a\right), \dot{q}_a\left(t_a\right)\right], \tag{5}$$

where the dot denotes the derivative with respect to the time parameter t_a corresponding to the particle a. An example of such dynamical system gives the Fokker theory of direct electromagnetic interaction of two charges described by the action (the velocity of light is taken equal unity) [1]:

$$\begin{aligned} I_F &= -m_1 \int ds_1 - m_1 \int ds_2 \\ &+ \frac{1}{2} e_1 e_2 \int dx_1^{\mu} \int dx_{2\mu} \delta\left(s_{12}^2\right). \end{aligned} \tag{6}$$

Here $x_{1,2}^{\mu}, \mu = 0, 1, 2, 3$ are coordinates of charges in the Minkowsky space, indices are lowered and rised by means of the metric $\eta_{\mu\nu} \equiv diag\left(+1, -1, -1, -1\right)$, and short notations for squares and scalar products of vectors are

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used:

$$
\begin{aligned}
ds_{1,2}^2 &\equiv dx_{1,2}^{\mu} dx_{1,2\mu}, \\
s_{1,2}^2 &\equiv (x_1^{\mu} - x_2^{\mu})(x_{1\mu} - x_{2\mu}) \equiv (x_1 - x_2)^2 .
\end{aligned}
$$

If we take $x_{1,2}^0 \equiv t_{1,2}$, and $\left\{x_{1,2}^i, i = 1, 2, 3\right\} \equiv q_{1,2}$, the Fokker action (6) may be written in the form (5):

$$
\begin{aligned}
I &= -m_1 \int_0^{T_1} \sqrt{1 - \dot{q}_1^2}\, dt_1 - m_2 \int_0^{T_2} \sqrt{1 - \dot{q}_2^2}\, dt_2 \qquad (7) \\
&+ \frac{1}{2} \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 \delta\left((t_1 - t_2)^2 - (q_1 - q_2)^2\right)\left(1 - \dot{q}_1 \dot{q}_2\right).
\end{aligned}
$$

The multi-time dynamics of the system, which follows from the stationarity condition for the action (3), is determined by the system of Euler-Lagrange (EL) equations for each particle in its own time parameter t_a:

$$
\frac{\delta I}{\delta q_a} - \frac{d}{dt_a} \frac{\delta I}{\delta \dot{q}_a} = 0 \qquad (8)
$$

In the case of Fokker theory with the action (6), the set of equations (8) is reduced to two ($3D$-vector) integro-differential equations.

III. GENERALIZED CANONICAL FORM OF MULTI-TIME DYNAMICAL THEORY

Let us define canonical momenta of particles as variational derivatives of the action (5) with respect to velocities:

$$
p_a(t_a) \equiv \frac{\delta I}{\delta \dot{q}_a(t_a)}. \qquad (9)
$$

If the action has the Lagrangian form (1), this definition is equivalent to the ordinary definition of canonical momenta. Let us assume that the set of equations (9) can be solved with respect to velocities, and the solution is:

$$
\dot{q}_a(t_a) = F_a\left(t_a; [q_a(t_a), p_a(t_a)]\right). \qquad (10)
$$

This solution is a function of a corresponding time parameter and a functional of a trajectory $(q_a(t_a), p_a(t_a))$ of the system in its phase space. The following step in our approach differs from a similar step in the ordinary canonical theory. Let us define a generalized Hamiltonian by means of generalized Legendre transformation as follows:

$$
\begin{aligned}
H[q_a(t_a), p_a(t_a)] \equiv & \left[\sum_a \int_0^{T_a} dt_a p_a(t_a) \dot{q}_a(t_a)\right. \\
& \left.\left. -I\left[q_a(t_a), \dot{q}_a(t_a)\right]\right]\right|_{\dot{q}=F}. \qquad (11)
\end{aligned}
$$

Then the action may be written in a generalized canonical form:

$$
\begin{aligned}
I[q_a(t_a), p_a(t_a)] =& \sum_a \int_0^{T_a} dt_a p_a(t_a) \dot{q}_a(t_a) \qquad (12) \\
& -H[q_a(t_a), p_a(t_a)].
\end{aligned}
$$

Let us check, first of all, that the condition of stationarity of the action (12) with respect to independent variations of coordinates $q_a(t_a)$ of particles, and their momenta $p_a(t_a)$, is equivalent to the system of EL equations (8). It is usefull to introduce at this stage a generalized Poisson brackets defined by the canonical commutational relations (all other brackets are zero):

$$
\left\{q_{ai}(t_a), p_{ak}\left(\widetilde{t}_a\right)\right\} = \delta_{ik} \delta\left(t_a - \widetilde{t}_a\right). \qquad (13)
$$

"Inner" indices of the canonical variables are written here in the open form. The stationarity condition for the action (12) can be written with the use of these brackets as a system of generalized canonical relations:

$$
\{q_a(t_a), I\} = \{p_a(t_a), I\} = 0. \qquad (14)
$$

Here, the first bracket equals is zero according to the following equalities:

$$
\begin{aligned}
\{q_a(t_a), I\} &= \dot{q}_a(t_a) - \{q_a(t_a), H\} = \dot{q}_a(t_a) \\
& - \left[\dot{q}_a(t_a) + \sum_b \int_0^{T_b} dt_b p_b(t_b) \frac{\delta F_b}{\delta p_a(t_a)}\right. \\
& \left. - \sum_b \int_0^{T_b} dt_b \frac{\delta I}{\delta \dot{q}_b(t_b)} \frac{\delta F_b}{\delta p_a(t_a)}\right] \\
&= 0, \qquad (15)
\end{aligned}
$$

where the definition of momenta (9) and a solution of these equalities with respect to velocities (10) were used. The second bracket in Eq. (14) also equals to zero due to the EL equations (8):

$$
\begin{aligned}
\{p_a(t_a), I\} &= \dot{p}_a(t_a) - \{p_a(t_a), H\} = \dot{p}_a(t_a) \\
& - \left[-\sum_b \int_0^{T_b} dt_b p_b(t_b) \frac{\delta F_b}{\delta q_a(t_a)}\right. \\
& + \sum_b \int_0^{T_b} dt_b \frac{\delta I}{\delta \dot{q}_b(t_b)} \frac{\delta F_b}{\delta q_a(t_a)} \\
& \left. + \frac{\delta I}{\delta q_a(t_a)}\right] \\
&= 0. \qquad (16)
\end{aligned}
$$

Therefore, the stationarity condition for the generalized canonical form of the action (12) is equivalent to the original equations of motion (8) of multi-time dynamical system.

IV. GENERALIZED CANONICAL FORM OF THE FOKKER THEORY

In the Fokker theory, equations (9) form the set of integral equations with respect to velocities:

$$\begin{aligned} p_{1i}(t_1) &= -\frac{m_1\dot{q}_{1i}(t_1)}{\sqrt{1-\dot{q}_1^2}} \\ &\quad -\frac{1}{2}e_1e_2\int_0^{T_2} dt_2\dot{q}_{2i}(t_2)\,\delta\left(s_{12}^2\right), \\ p_{2i}(t_2) &= -\frac{m_2\dot{q}_{2i}(t_2)}{\sqrt{1-\dot{q}_2^2}} \\ &\quad -\frac{1}{2}e_1e_2\int_0^{T_1} dt_1\dot{q}_{1i}(t_1)\,\delta\left(s_{12}^2\right). \end{aligned} \tag{17}$$

One can solve this set by use of the perturbation theory with respect to the parameter e_1e_2 of the direct interaction. In the first order in the parameter e_1e_2 we have:

$$\begin{aligned} \dot{q}_{1i}(t_1) &= -\frac{p_{1i}(t_1)}{\sqrt{p_1^2+m_1^2}} \\ &\quad +\frac{1}{2}\int_0^{T_2} dt_2\frac{e_1e_2\delta\left(s_{12}^2\right)}{\sqrt{p_1^2+m_1^2}\sqrt{p_2^2+m_2^2}} \\ &\quad \times\left[p_{1i}\frac{p_1p_2}{p_1^2+m_1^2}-p_{2i}\right], \\ \dot{q}_{2i}(t_2) &= -\frac{p_{2i}(t_2)}{\sqrt{p_2^2+m_2^2}} \\ &\quad +\frac{1}{2}\int_0^{T_1} dt_1\frac{e_1e_2\delta\left(s_{12}^2\right)}{\sqrt{p_1^2+m_1^2}\sqrt{p_2^2+m_2^2}} \\ &\quad \times\left[p_{2i}\frac{p_1p_2}{p_2^2+m_2^2}-p_{1i}\right]. \end{aligned} \tag{18}$$

In the first order in the parameter e_1e_2 the generalized Hamiltonian defined by Eq. (11) is

$$\begin{aligned} &H\left[p_1(t_1), q_1(t_1), p_2(t_2), q_2(t_2)\right] \\ &= \int_0^{T_1}\sqrt{p_1^2+m_1^2}dt_1+\int_0^{T_2}\sqrt{p_2^2+m_2^2}dt_2 \\ &\quad +\frac{e_1e_2}{2}\int_0^{T_1}dt_1\int_0^{T_2}dt_2\delta\left(s_{12}^2\right) \\ &\quad \times\left(1-\frac{p_1p_2}{\sqrt{p_1^2+m_1^2}\sqrt{p_2^2+m_2^2}}\right) \end{aligned} \tag{19}$$

The first two terms in the right hand side of Eq. (19) are Hamiltonian parts of the canonical action of free particles and two parts of the third term describe, in the first order in e_1e_2, their Coulomb and magnetic interactions.

V. QUANTUM ACTION PRINCIPLE

Using the generalized canonical form of the action of the multi-time dynamical theory, we can perform a modified canonical quantization procedure proposed in the work [2]. The central object in the modified quantum theory is an action operator $\widehat{I}$ defined in a space of wave functionals. In turn, a wave functional $\Psi\left[q_a(t_a)\right]$ is a functional of a trajectory $q_a(t_a)$ of the system with fixed end points in the configuration space. To define the action operator, we represent the canonical variables as operators in the space of wave functionals as follows:

$$\begin{aligned} \widehat{q}_{ak}(t_a)\Psi &\equiv q_{ak}(t_a)\Psi, \\ \widehat{p}_{ak}(t_a)\Psi &\equiv \frac{\widetilde{\hbar}}{i}\frac{\delta\Psi}{\delta q_{ak}(t_a)}. \end{aligned} \tag{20}$$

Here we write in the open form the "inner" indice $k=1,2,...,n$. The constant $\widetilde{\hbar}$ differs from the "ordinary" Plank constant $\hbar$, in particular, its physical dimensionality is $\left[\widetilde{\hbar}\right]=Joule\cdot s^2$. A relationship between two constants can be established after a determination of observables in the proposed quantum theory and comparison between the theory and an experiment. The operators (20) are formally Hermitian with respect to the scalar product in the space of wave functionals:

$$\begin{aligned} &(\Psi_1,\Psi_2) \\ &= \int\prod_a\prod_{t_a}d^nq_a(t_a)\,\overline{\Psi}_1\left[q_a(t_a)\right]\Psi_2\left[q_a(t_a)\right]. \end{aligned} \tag{21}$$

A wave functional $\Psi\left[q_a(t_a)\right]$ has now the natural probabilistic interpretation: $\left|\Psi\left[q_a(t_a)\right]\right|^2$ is a probability density of that a system moves along a trajectory in a neighbourhood of given trajectory $q_a(t_a)$. The operators (20) obey the following permutation relations (all other commutators are zero):

$$\left[\widehat{q}_{ai}(t_a), \widehat{p}_{ak}\left(\widetilde{t}_a\right)\right]=i\widetilde{\hbar}\delta_{ik}\delta\left(t_a-\widetilde{t}_a\right). \tag{22}$$

Replacing canonical variables in the action (12) by their operator realization (18), one obtains an action operator $\widehat{I}$. This operator is also formally Hermitian with respect to the scalar product (21).

Now, let us return to the formulation of the quantum action principle. For the action operator $\widehat{I}$ we consider the eigenvalue problem:

$$\widehat{I}\Psi=\Lambda\Psi. \tag{23}$$

In a general case the set of eigenvalues Λ is parametrized by a set of parameters which form a smooth manifold. The stationarity condition for this smooth function of free parameters forms the content of the quantum action principle. To make this formulation more concrete, let us consider the Fokker theory. In this case the action

operator is

$$\widehat{I} = \int_0^{T_1} dt_1 \dot{q}_{1k}(t_1) \frac{\widetilde{\hbar}}{i} \frac{\delta}{\delta q_{1k}(t_1)} + \int_0^{T_2} dt_2 \dot{q}_{2k}(t_2) \frac{\widetilde{\hbar}}{i} \frac{\delta}{\delta q_{2k}(t_2)} - \widehat{H}, \tag{24}$$

where the operator $\widehat{H}$ is obtained by the substitution of the operators (20) into the generalized Hamiltonian of the Fokker theory. In the first order in $e_1 e_2$, the latter one is given by the expression (21). Even in this approximation this expression is too complex due to the presence of square routs, in particular, in the denominator of the last term. To simplify the subsequent consideration let us consider a non-relativistic limit of the theory. In this limit we have:

$$\widehat{H} = -\int_0^{T_1} dt_1 \frac{\widetilde{\hbar}^2}{2m_1} \frac{\delta^2}{\delta q_1^2(t_1)} - \int_0^{T_2} dt_2 \frac{\widetilde{\hbar}^2}{2m_2} \frac{\delta^2}{\delta q_2^2(t_2)} + \frac{1}{2} e_1 e_2 \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 \delta\left(s_{12}^2\right). \tag{25}$$

Here, the last term in the right hand side describes with the relativistic accuracy the Coulomb interaction of charges. The problem is that this term is a non-analytic functional of trajectories of charges. In the case of analytic potentials, a concrete formulation of the action principle was proposed in the preprint [3]. To use this formulation, let us regularize the Coulomb potential in Eq. (25), replacing the δ-function by the exponent:

$$\frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{\left(s_{12}^2\right)^2}{2\sigma}\right). \tag{26}$$

At the final stage of calculations the limit $\sigma \to 0$ is assumed. After this regularization the interaction term becomes an analytic functional of trajectories of charges, and the formulation of the quantum action principle proposed in [3] is also applicable in the case under consideration. Formulation of the quantum action principle in general case will depend, in particular, on the definition of the square route of the functional-differential operator:

$$\sqrt{-\widetilde{\hbar}^2 \frac{\delta^2}{\delta q^2(t)} + m^2}. \tag{27}$$

This will be a subject of a subsequent work.

VI. CONCLUSIONS

In conclusion, in the present paper we demonstrated that the quantum action principle gives a possibility to formulate a correct quantum version of multi-time dynamical theory with a proper probabilistic interpretation.

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